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On the Sum of the Reciprocals of Squares

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Abstract

The Fourier Series has always been a great tool by turning the most complicated functions into a simple approximation by several sine and cosine terms. However, given the fact that its called an approximation arouses the question of how accurate is this approximation, especially with functions that have a corner at x values. In this project, we will show how the series that is the sum of the reciprocals of squares arises in the study of Fourier series. We will discuss the convergence and the rate of convergence of this series. This result arose during exploration on mathematical software Sage. We will indicate some direction for future research.

Background

In the 1700s, Daniel Bernoulli, and Leonard Euler used these series to solve problems related to vibrating strings and astronomy. Moreover, French mathematician Joseph Fourier used these series in 1822 to solve problems in heat conduction[1].

Introduction

As mentioned before, the Fourier series is a method used by scientists, mathematicians, and engineers to approximate complicated functions in terms of sine and cosine terms. The series is described as:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n * cos(nx) + b_n * \sin(nx)$$

Where: $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$ $\quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) * \cos(nx)dx$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) * \sin(nx)\, dx$[2] $\quad f(x)\ is\ the\ function\ of\ interest$

$a_n\ and\ b_n\ are\ called\ the\ Fourier\ coefficients, and\ a_0\ represents\ twice\ the\ average\ area\ of\ the\ function$

Gibb's Phenomenon

Mathematicians describe this phenomenon when dealing with functions that have a corner at a certain x value where at this point, the derivative of the function is not continuous. This phenomenon states when approximating the Fourier series of that function, there is an overshoot at that corner, and the height of the Fourier series is approximately 8.99% of the height at the overshoot[3]. This is shown in Figure 1 to the right.

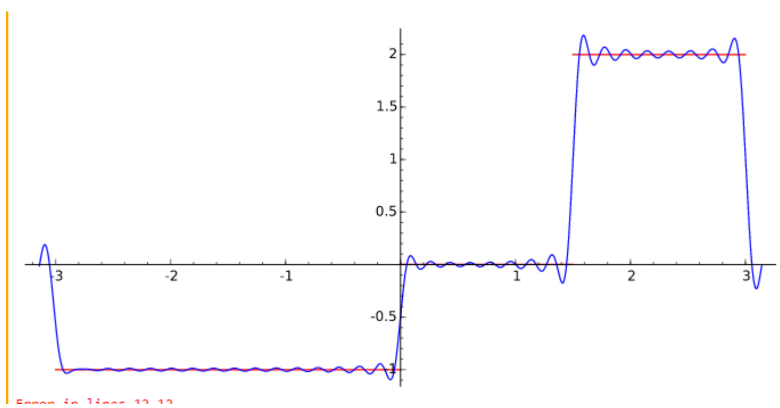


Figure 1 shows the Fourier series of a function that has jumps of discontinuity at different x values

Fourier Series

I have been using Sage mathematical software to simulate several graphs, and their respective Fourier series at different number of terms. In Sage, there is a built in function to compute the Fourier series of a function. However, I wrote my own code to simulate the Fourier series by simulating the above equations found in the Introduction. The reason why I used my own code for the Fourier series was even though my own code was bigger, but it was able to compute more terms than the built in function. I first used the built in function, and sometimes the computer freezes when it is prompted to compute 500 terms. But when I used my own code, I was able to get the Fourier series of functions up to 2000 terms at a faster rate.

The Fourier series is a lot similar to the Taylor series. In the Taylor series, the first few terms (first, second, and third derivative) do not clearly add up to the function of interest. However, as you

$$f(x) = \begin{cases} -\pi x - x^2 & -\pi < x < 0 \\ \pi x - x^2 & 0 < x < \pi \end{cases}$$

I used Sage to simulate the plot of this function. The plot is found in Figure 2 below

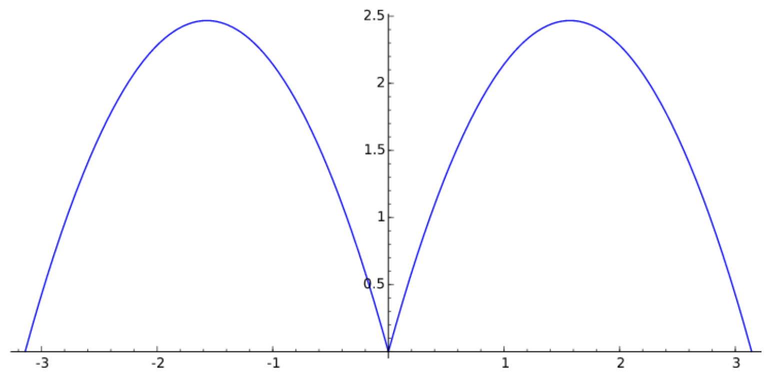


Figure 2 shows the plot of the function of interest. This function represents two humps symmetrical with respect to the y-axis.

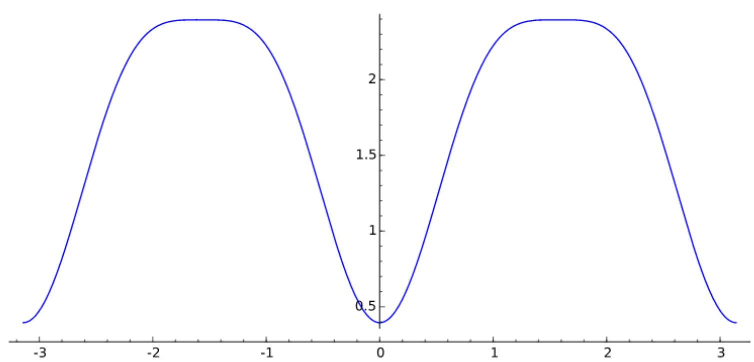


Figure 3 shows the Fourier series of the function at n = 5 terms

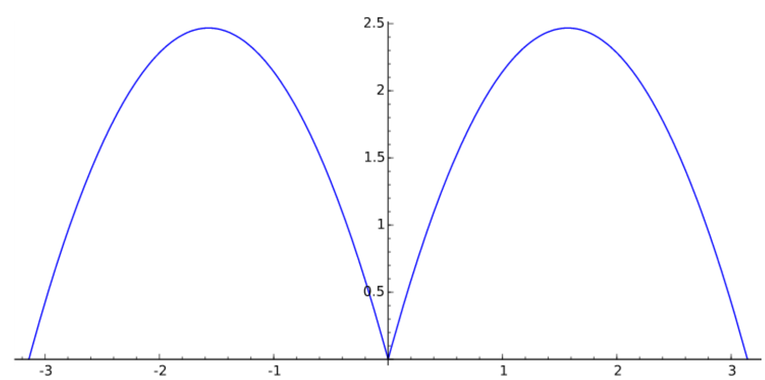


Figure 4 shows the Fourier series at n = 500 terms

The Fourier series is an approximation as mentioned in the abstract, so I had to plot the difference between the original function, and the Fourier series function.

The reason why I plot this difference is to check how much do they differ, and what peak height at the corners (at x = 0) does each difference function generate with respect to the number of terms the Fourier series function is plotted on. Figures 5 and 6 shows the difference at different number of terms for the Fourier series of the graph.

Notice how the number of terms got bigger, it gave a better approximation of the function. You can tell this by eyeballing at both the original and Fourier series functions and tell that they look alike. To be more specific, if you plot the difference function at a higher number of terms, you notice that the peak height at the corners gets less every time. I wanted to check if there is something analogous with Gibb's

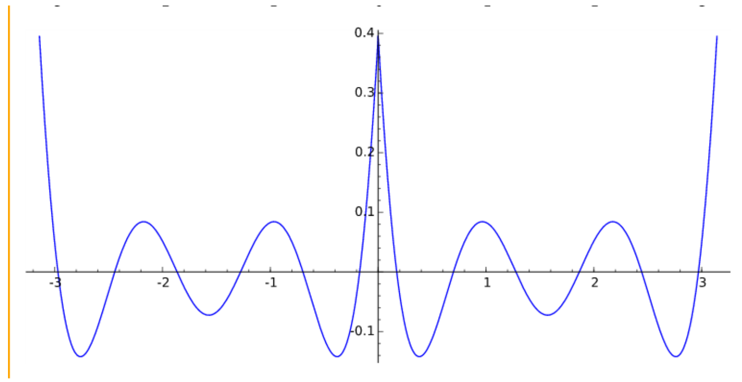


Figure 5 shows the difference at n = 5 terms

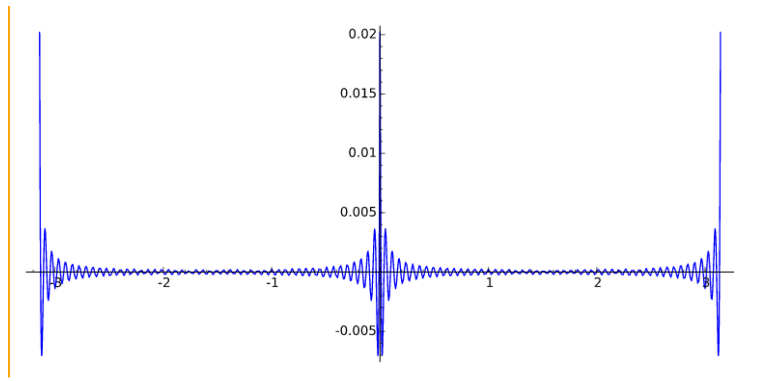
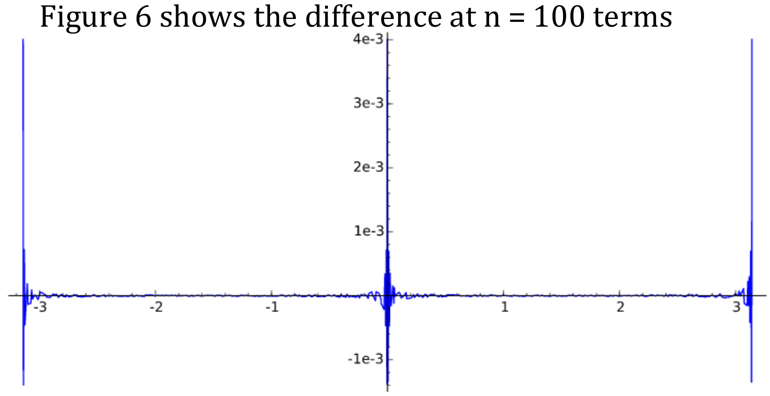


Figure 6 shows the difference at n = 100 terms



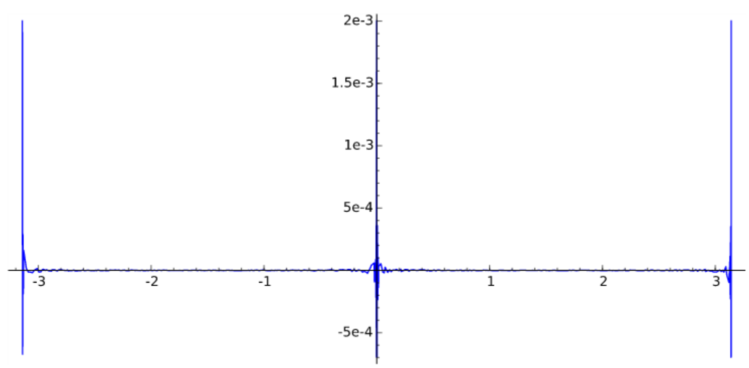


Figure 8 shows the difference at n = 1000 terms

Numerical Analysis

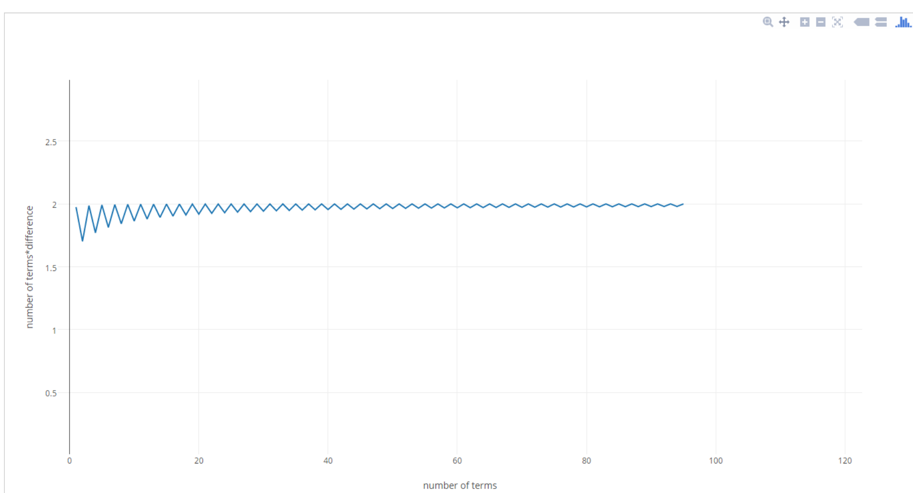
I approached the problem with two different methods, and try to get to the same result. The first method is the numerical analysis. I used Sage code to give me a table of data shown in Figure 9. I wrote this code, because when I plotted the difference, it was difficult to eyeball and try to find the values at the peak (at x = 0).

Notice how as the number of terms increase, the product of the number of terms and the difference of the functions is approaching the number 2. We can say that this product is converging to the number 2 in the long run.

Term	Difference	Term*Difference
48	0.0425467743683365	1.99969839531182
50	0.0408106632572254	1.99972249960404
52	0.0392106632572253	1.99974382611849
54	0.0377313733163969	1.99976278576903
56	0.0363596312039141	1.99977971621527
58	0.0350841209998325	1.99979489699045
60	0.0338950603577397	1.99980856110664
62	0.0327839492466286	1.99982090404435
64	0.0317433665203017	1.99983209077901
66	0.0307668040203017	1.99984226131961
68	0.0298485303747555	1.99985153510862
70	0.0289834784716412	1.99986001454324
72	0.0281671519410289	1.99986778781305
74	0.0273955470027574	1.99987493120129
76	0.0266650868128377	1.99988151096283
78	0.0259725660372145	1.99988758486552
80	0.0253151038412909	1.99989320346198
82	0.0246901038412908	1.99989841114455
84	0.0240952198436704	1.99990324702464
86	0.0235283264196340	1.99990774566889
88	0.0229874935369947	1.99991193771854
90	0.0224709646113750	1.99991585041237
92	0.0219771374508810	1.99991950803017
94	0.0215045476588207	1.99992293227033
96	0.0210518541323381	1.99992614257212
98	0.0206178263545602	1.99992915639234
100	0.0202013332266968	1.99993198944298

Figure 9 shows a table of the number of term, the difference between original and Fourier series functions, and their product

There is much online free software that allows you to plot your data. I chose software called Plotly. One of the reasons is that it's free, and also it can graph many inputs at the same time. In figure 10, I made this plot from 100 terms.



Algebraic Analysis

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_0 = \frac{2}{\pi}\int_{0}^{\pi} (\pi x - x^2)dx$$

$$a_0 = \frac{2}{\pi}\left(\frac{\pi x^2}{2} - \frac{x^3}{3}\right)\Big|_{0}^{\pi}$$

$$a_0 = \frac{2}{\pi}\left(\frac{\pi^3}{2} - \frac{\pi^3}{3}\right) = \frac{2}{\pi}\left(\frac{\pi^3}{6}\right) = \frac{\pi^2}{3}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) * \sin(nx)dx$$

$b_n = 0$ by symmetry

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) * \cos(nx)dx$$

$$a_n = \frac{1}{\pi}\left(\int_{-\pi}^{0} f(x) * \cos(nx)\,dx + \int_{0}^{\pi} f(x) * \cos(nx)\,dx\right)$$

$$a_n = \frac{1}{\pi}\left(\int_{-\pi}^{0}(-\pi x^2 - x^2) * \cos(nx)\,dx + \int_{0}^{\pi}(\pi x - x^2) * \cos(nx)\,dx\right)$$

$$a_n = \frac{2}{n^2}(-\cos(n\pi) - 1)$$

Where n is the number of terms

Notice the 2 cases:
If n is odd, $a_n = 0$
If n is even, $a_n = \frac{-4}{n^2}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n * cos(nx) + b_n * \sin(nx)$$

$$f(x) = \frac{\pi^2}{6} - \sum_{\substack{n=2 \\ even\ n's}}^{\infty}\left(\frac{-4}{n^2}\right)\cos(nx)$$

Let n = 2m

$$f(x) = \frac{\pi^2}{6} - \sum_{m=1}^{\infty}\left(\frac{4}{4m^2}\right)\cos(2mx)$$

$$f(x) = \frac{\pi^2}{6} - \sum_{m=1}^{\infty}\left(\frac{1}{m^2}\right)\cos(2mx)$$

$At\ x = 0,$

$$f(0) = \frac{\pi^2}{6} - \sum_{m=1}^{\infty}\left(\frac{1}{m^2}\right)\cos(0)$$

$$f(0) = \frac{\pi^2}{6} - \sum_{m=1}^{\infty}\left(\frac{1}{m^2}\right)$$

At $x = 0, we\ know\ f(0) = 0$

$$\frac{\pi^2}{6} - \sum_{m=1}^{\infty}\left(\frac{1}{m^2}\right) = 0$$

$$\sum_{m=1}^{\infty}\left(\frac{1}{m^2}\right) = \frac{\pi^2}{6}$$

The Basel Problem

Notice how the Fourier series ends up with the sum of reciprocals of squares which arises the Basel Problem. The Basel Problem basically asks about the exact sum of the reciprocals of squares. At infinity, it is found that this sum is equal to $\frac{\pi}{6}$ as approved by Einstein[4], and the Fourier series.

However, in this project as I use to Sage to simulate the Fourier series, I always simulate to give a finite number of terms (even if the number of terms is very big, it is still less than infinity.

So, what does $\sum_{m=1}^{N}\left(\frac{1}{m^2}\right)$ equal?

We know that at infinity, the sum of reciprocals of squares approaches pi/6, so my assumption was that

$$\sum_{m=1}^{N}\left(\frac{1}{m^2}\right) \approx \frac{\pi^2}{6} - \frac{1}{N}$$

$$f(0) = \frac{\pi^2}{6} - \sum_{m=1}^{N}\left(\frac{1}{m^2}\right)$$

$$\frac{\pi^2}{6} - \sum_{m=1}^{N}\left(\frac{1}{m^2}\right) = 0$$

$$f(0) = \frac{1}{N}$$

Recall that we were using n = 2m, so the number of terms to multiply will actually by 2N instead of N

$$2N * f(0) = 2$$

Conclusion

The product of term and difference of functions converged to a specific number.

$$\boldsymbol{\sum_{m=1}^{N}\left(\frac{1}{m^2}\right) \approx \frac{\pi^2}{6} - \frac{1}{N}}$$

I am interested in checking the second term of the sum of the inverse of squares (if there is such). This is known as the asymptotic expansion.

$$\sum_{m=1}^{N}\left(\frac{1}{m^2}\right) \approx \frac{\pi^2}{6} - \frac{1}{N} - \frac{k?}{N^{2?}}$$

I am also interested if this idea would apply to other functions that have a corner.

Bibliography

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2. Tolstov, G. P., and Richard A. Silverman. *Fourier Series*. New York: Dover Publications [etc.] ;, 1976. Print.

3. Wikipedia,. 'Gibbs Phenomenon'. N.p., 2015. Web. 25 June 2015.

4. Wikipedia,. 'Basel Problem'. N.p., 2015. Web. 25 June 2015.