Spring 2019

Poisson's Equation

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By Ehren Braun

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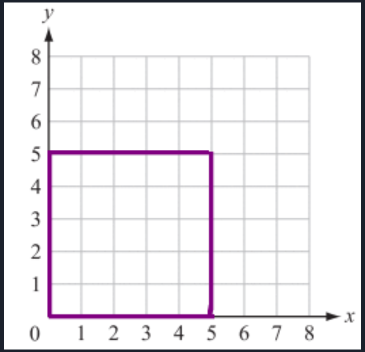
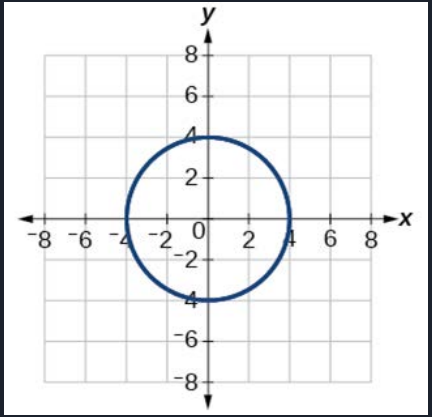
- Time-independent (Steady State)
- Geometry determines Δ





Poisson’s Equation Example

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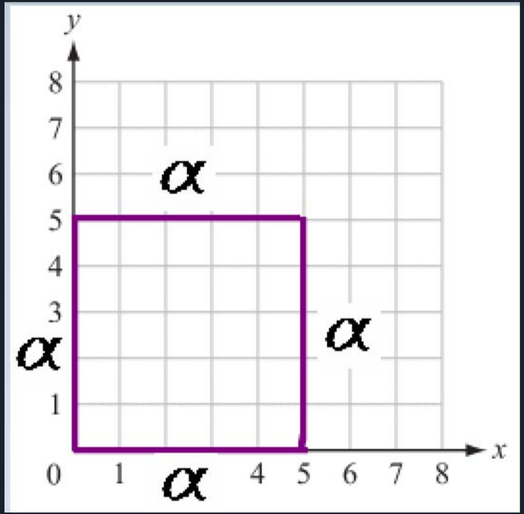
- Rectangular Plate
  - Edges(boundary) given by u = α
    - α can vary



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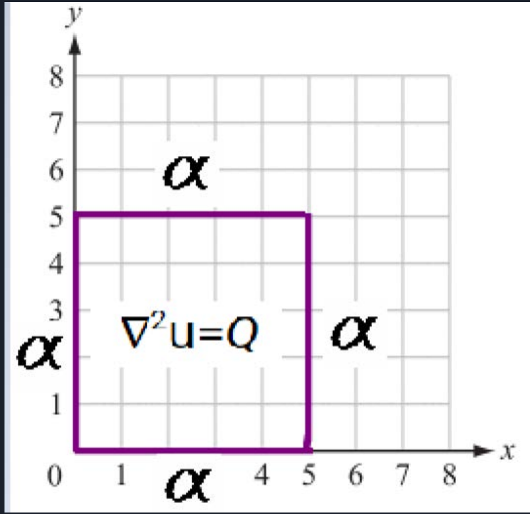

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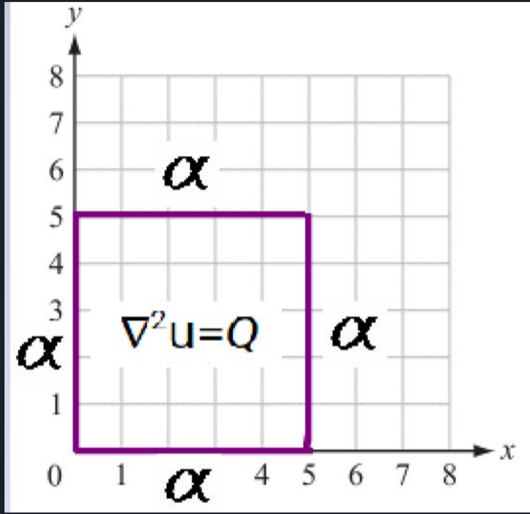

Poisson's Equation Example

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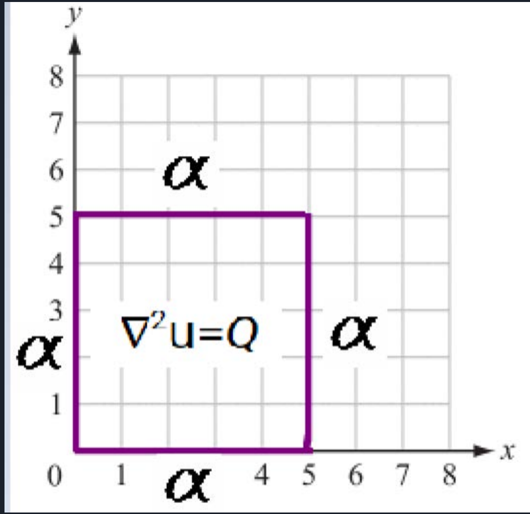
  - Edges(boundary) given by u = α
    - α can vary
  - Rest given by $\Delta u = Q$
- Nonhomogenous from Q and α
- Easier with homogenous components



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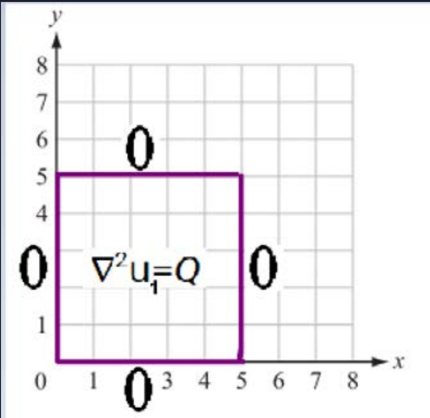

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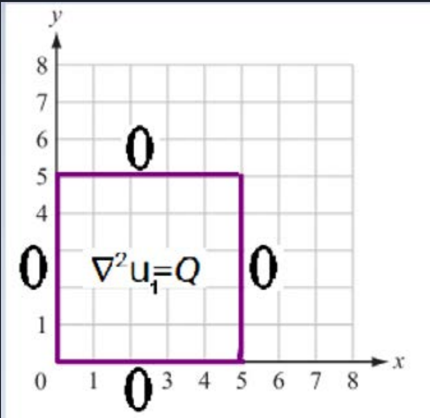

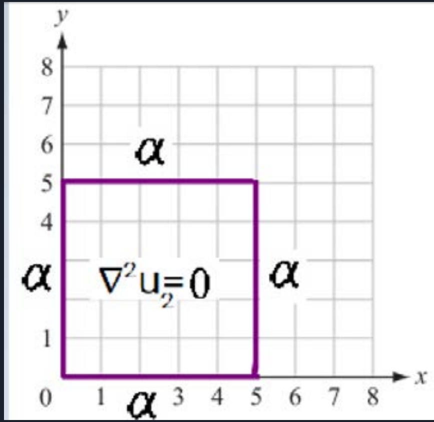

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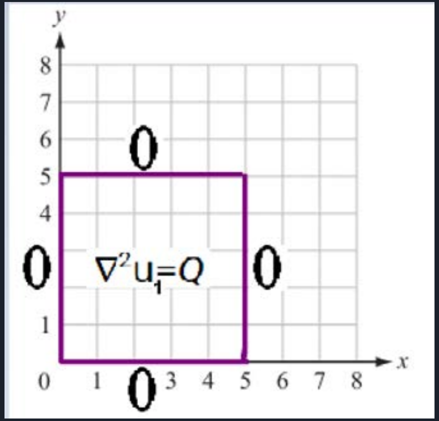

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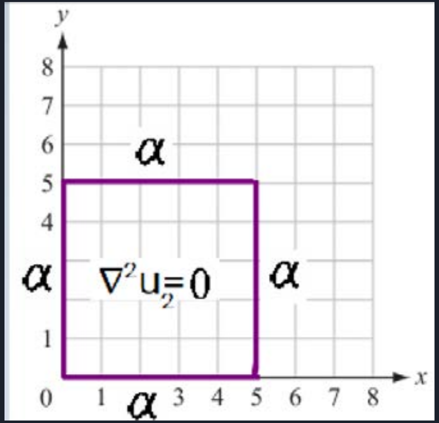

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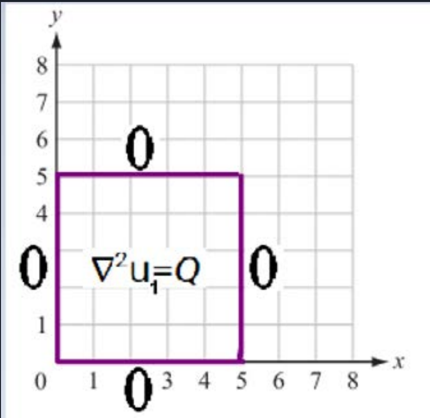
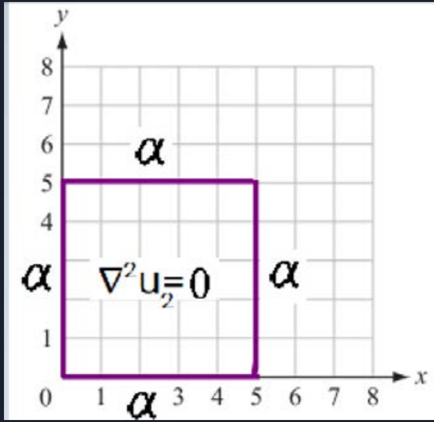
- Two “easier” problems to solve





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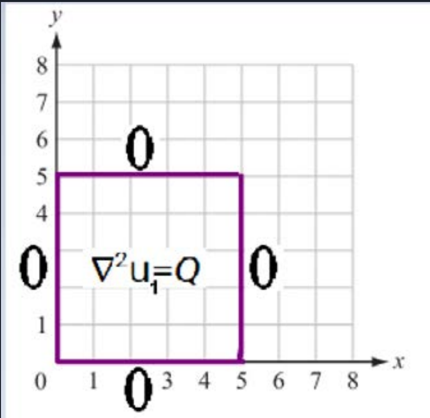

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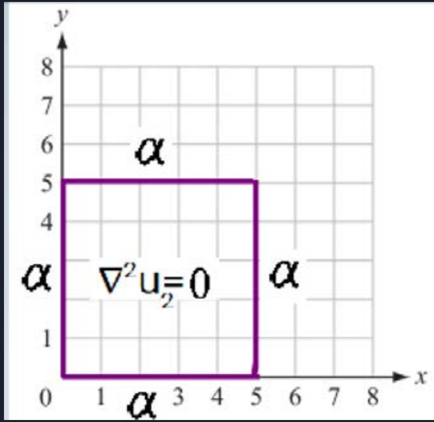
- Two “easier” problems to solve
- Similar for other regions



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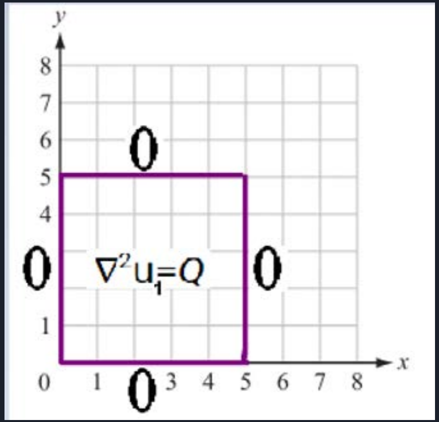
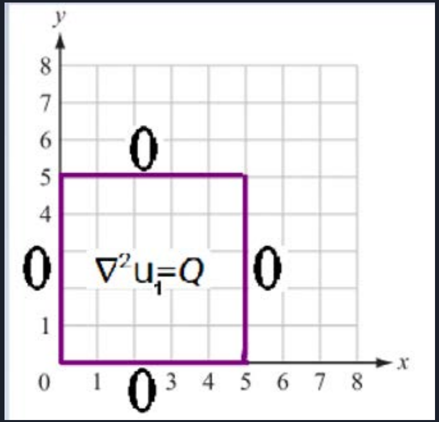


Solving $\Delta u_1 = Q$, $u_1 = 0$ on Boundary

- With homogeneous boundaries



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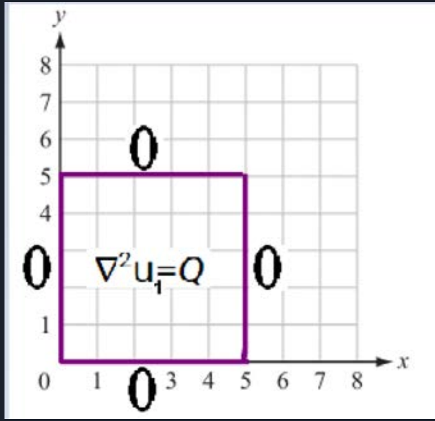

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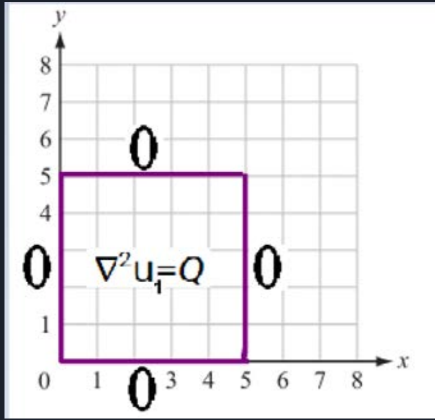

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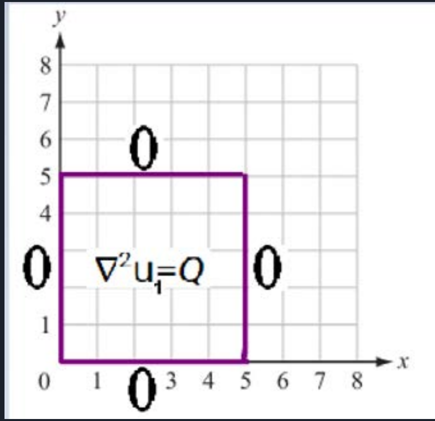
- Two different ways of Expansion
    - Eigenfunctions related to $\Delta u_1 = 0$



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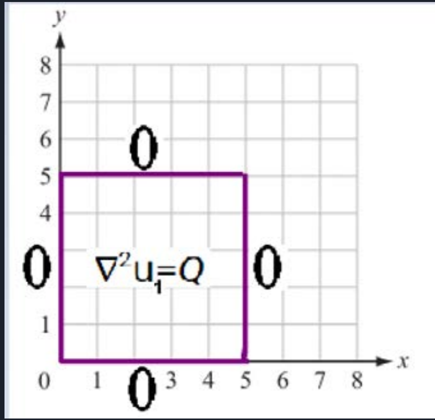
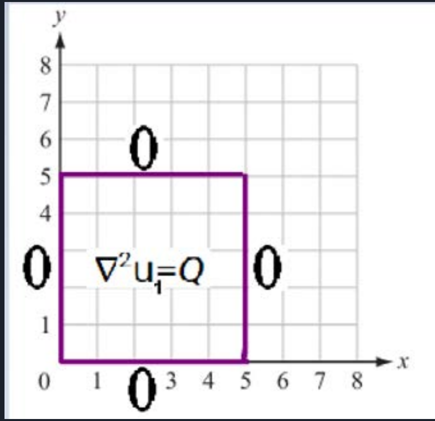


Solving $\Delta u_1 = Q$, $u_1 = 0$ on Boundary

- With homogeneous boundaries
- Implies eigenfunction expansion method
- Two different ways of Expansion
  - Eigenfunctions related to $\Delta u_1 = 0$
  - Eigenfunctions related to $\Delta \phi + \lambda \phi = 0$
- Methods are different, but related
  - One-dimensional vs Two-dimensional



One-Dimensional Eigenfunctions for u_1

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$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

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- $$\frac{X''}{X} = \frac{-Y''}{Y} = -\lambda$$

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- $X''Y + XY'' = 0$

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  - Note: Could subtract Xs instead

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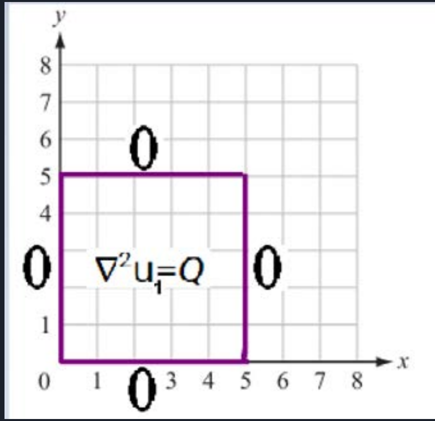
$$\frac{X''}{X} = \frac{-Y''}{Y} = -\lambda$$



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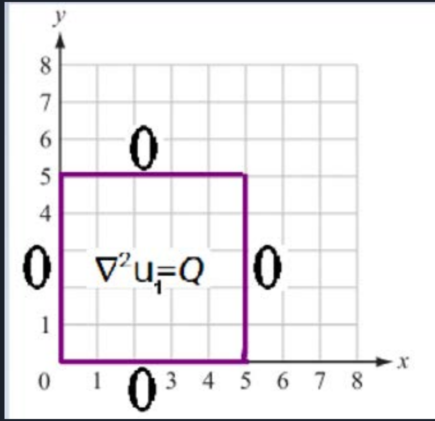

One-Dimensional Eigenfunctions for u_1

$$\frac{X''}{X} = \frac{-Y''}{Y} = -\lambda$$

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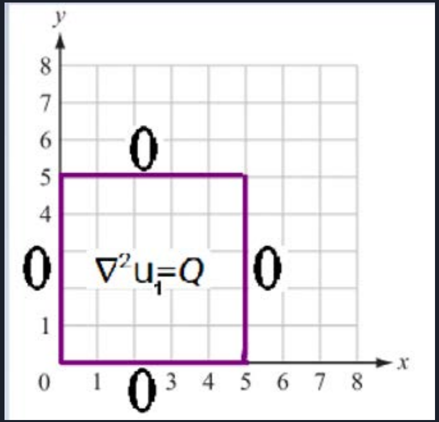
3 Cases: $\lambda > 0$ $\lambda < 0$ $\lambda = 0$,

Looking for Non-Trivial Solutions (Only $\lambda > 0$)



One-Dimensional Eigenfunctions for u_1

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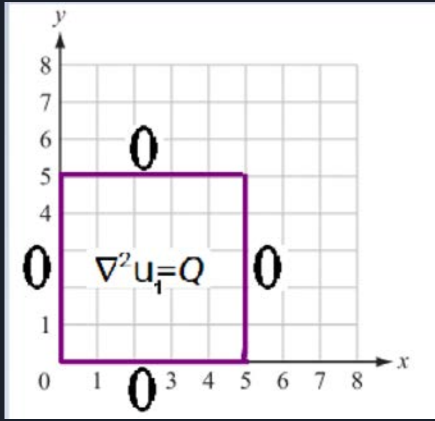
Looking for Non-Trivial Solutions (Only λ > 0)

$\lambda > 0$: $c_1 sin(\sqrt{\lambda}x) + c_2 cos(\sqrt{\lambda}x)$



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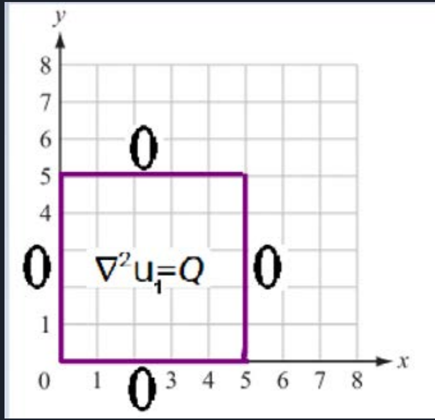

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Looking for Non-Trivial Solutions (Only λ > 0)

$\lambda > 0$: $c_1 sin(\sqrt{\lambda}x) + c_2 cos(\sqrt{\lambda}x)$

Boundary Conditions => $X_n = c_n sin(\frac{n\pi x}{L})$, $n = 1, 2, ...$

One-Dimensional Eigenfunctions for u_1

$$\frac{X''}{X} = \frac{-Y''}{Y} = -\lambda$$

Note:

$$\lambda = \left(\frac{n\pi}{L}\right)^2$$

$$n = 1, 2, \ldots$$

One-Dimensional Eigenfunctions for u_1

$$\frac{X''}{X} = \frac{-Y''}{Y} = -\lambda$$

$$Y'' = \lambda Y$$

Note:

$$\lambda = \left(\frac{n\pi}{L}\right)^2$$

$$n = 1, 2, \ldots$$

From Xs, $\lambda > 0$

One-Dimensional Eigenfunctions for u_1

$$\frac{X''}{X} = \frac{-Y''}{Y} = -\lambda$$

$$Y'' = \lambda Y$$

$$Y_n = a_n e^{\lambda y} + b_n e^{-\lambda y}$$

Note:

$$\lambda = \left(\frac{n\pi}{L}\right)^2$$

$$n = 1, 2, \ldots$$

One-Dimensional Eigenfunctions for u_1

$$\frac{X''}{X} = \frac{-Y''}{Y} = -\lambda$$

$$Y'' = \lambda Y$$

Note:

$$\lambda = \left(\frac{n\pi}{L}\right)^2$$

$$n = 1, 2, \ldots$$

$$Y_n = a_n e^{\lambda y} + b_n e^{-\lambda y}$$

Can be rewritten as:

$$Y_n = \widehat{a_n} sinh(\lambda y) + \widehat{b_n} cosh(\lambda y)$$

One-Dimensional Eigenfunctions for u_1

Now that we have X and Y:

One-Dimensional Eigenfunctions for u_1

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$$Y_n = a_n e^{\lambda y} + b_n e^{-\lambda y} \qquad \lambda = (\frac{n\pi}{L})^2$$

$$X_n = c_n sin(\frac{n\pi x}{L})$$

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For One-Dimensional: $u_1 = \sum_{n=1}^{\infty} b_n(y) sin(\frac{n\pi x}{L})$

One-Dimensional Eigenfunctions for u_1

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$$Y_n = a_n e^{\lambda y} + b_n e^{-\lambda y} \qquad \lambda = \left(\frac{n\pi}{L}\right)^2$$

$$X_n = c_n sin\left(\frac{n\pi x}{L}\right)$$

For One-Dimensional: $u_1 = \sum_{n=1}^{\infty} b_n(y) sin\left(\frac{n\pi x}{L}\right)$

Now apply the laplacian

One-Dimensional Eigenfunctions for u_1

Note: Q = Q(x, y)

$$u_1 = \sum_{n=1}^{\infty} b_n(y) sin(\frac{n\pi x}{L})$$

$$u_{1_{yy}} + u_{1_{xx}} = Q$$

One-Dimensional Eigenfunctions for u_1

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$$u_{1_{yy}} = \sum_{n=1}^{\infty} b_n(y)'' sin(\frac{n\pi x}{L})$$

$$u_{1_{xx}} = \sum_{n=1}^{\infty} -(\frac{n\pi}{L})^2 b_n(y) sin(\frac{n\pi x}{L})$$

One-Dimensional Eigenfunctions for u_1

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$$\sum_{n=1}^{\infty} [b_n(y)'' - (\frac{n\pi}{L})^2 b_n(y)] sin(\frac{n\pi x}{L}) = Q$$

One-Dimensional Eigenfunctions for u_1

Note: Q = Q(x, y)

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$$u_{1_{yy}} + u_{1_{xx}} = Q$$

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Now we just want only Ys

One-Dimensional Eigenfunctions for u_1

$$\sum_{n=1}^{\infty}\left[b_n(y)'' - \left(\frac{n\pi}{L}\right)^2 b_n(y)\right] sin\left(\frac{n\pi x}{L}\right) = Q$$

One-Dimensional Eigenfunctions for u_1

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$$\sum_{n=1}^{\infty}[b_n(y)'' - (\frac{n\pi}{L})^2 b_n(y)]\int_0^L sin(\frac{n\pi x}{L})sin(\frac{m\pi x}{L})dx = \int_0^L Qsin(\frac{m\pi x}{L})dx$$

Now we have 3 cases(Orthogonality): $m \neq n$, $m=n\neq 0$, $m=n=0$

One-Dimensional Eigenfunctions for u_1

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Doing these cases, only m=n≠0 is nonzero => $\int_0^L sin^2(\frac{n\pi x}{L}) dx = \frac{L}{2}$

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$$\sum_{n=1}^{\infty}[b_n(y)'' - (\frac{n\pi}{L})^2 b_n(y)] = \frac{2}{L}\int_0^L Q sin(\frac{n\pi x}{L})dx \equiv q_n(y)$$

$$Q = \sum_{n=1}^{\infty} q_n(y) sin(\frac{n\pi x}{L})$$

One-Dimensional Eigenfunctions for u_1

All there is left is $b_n(y)$

$$Y_n = \widehat{a_n} sinh(\lambda y) + \widehat{b_n} cosh(\lambda y)$$

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$$\sum_{n=1}^{\infty} [b_n(y)'' - (\frac{n\pi}{L})^2 b_n(y)] = \frac{2}{L} \int_0^L Q sin(\frac{n\pi x}{L}) dx \equiv q_n(y)$$

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One-Dimensional Eigenfunctions for u_1

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$$b_n(y) = sinh(\frac{n\pi[H-y]}{L})\int_0^y q_n(\xi) sinh(\frac{n\pi\xi}{L}) d\xi + sinh(\frac{n\pi y}{L})\int_y^H q_n(\xi) sinh(\frac{n\pi(H-\xi)}{L}) d\xi$$

Finishing One-Dimensional Eigenfunctions

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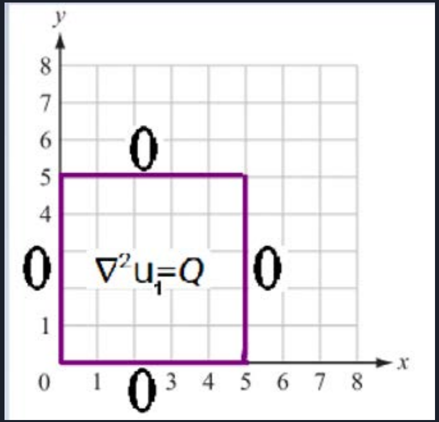
Two-Dimensional Eigenfunctions are easier

Two-Dimensional Eigenfunctions for u_1

- Relating to $\Delta\phi + \lambda\phi = 0$



Two-Dimensional Eigenfunctions for u_1

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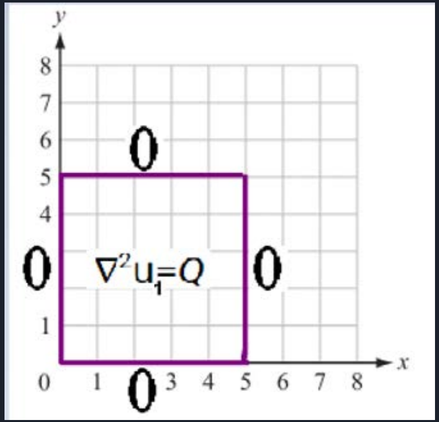

Two-Dimensional Eigenfunctions for u_1

- Relating to $\Delta\phi + \lambda\phi = 0$
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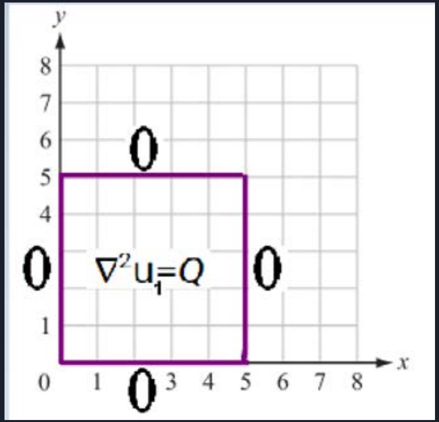

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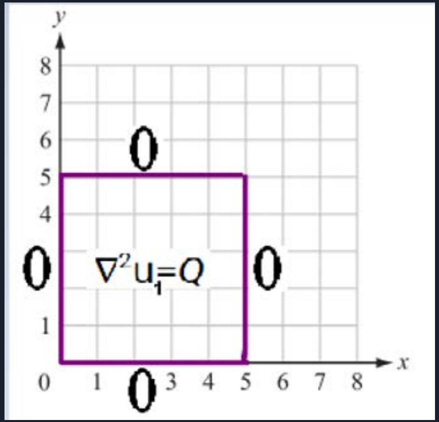

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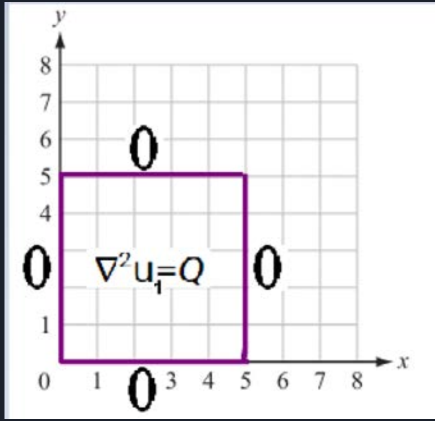

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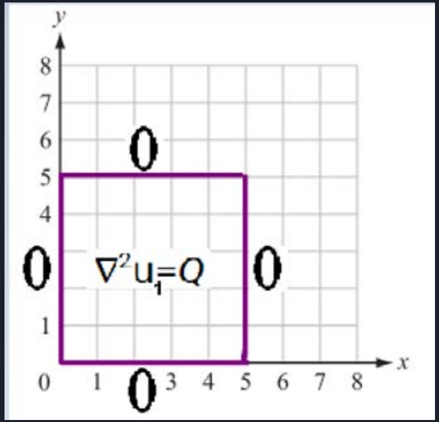

- Relating to $\Delta\phi + \lambda\phi = 0$
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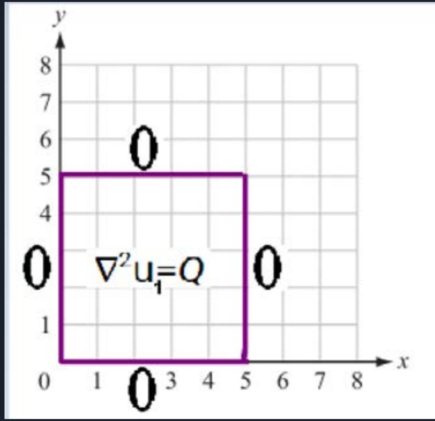
- $X''Y + XY'' = -\lambda XY$
- $\frac{X''}{X} + \frac{Y''}{Y} = -\lambda$
- Each term should be constant (λ not dependent)

Two-Dimensional Eigenfunctions for u_1



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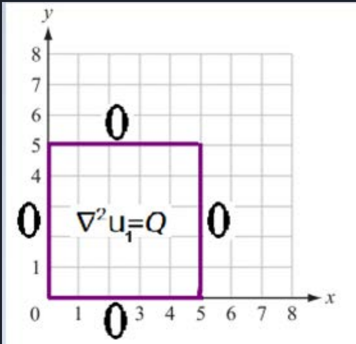
- Each term should be constant (λ not dependent)
- Let $\lambda=\lambda_x+\lambda_y$

Two-Dimensional Eigenfunctions for u_1

$$\frac{X''}{X} + \frac{Y''}{Y} = -(\lambda_x + \lambda_y)$$



Two-Dimensional Eigenfunctions for u_1

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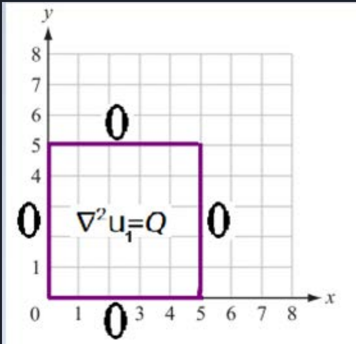
$$X'' = -X\lambda_x \qquad Y'' = -Y\lambda_y$$



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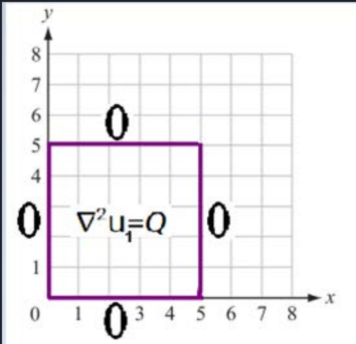

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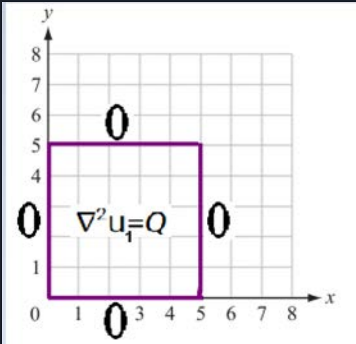

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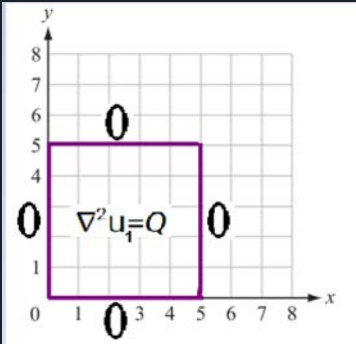
Boundary Conditions => $X_n = b_n sin(\frac{n\pi x}{L})$ $\quad \lambda_{x_n} = (\frac{n\pi}{L})^2$

$n = 1, 2, \ldots$

Two-Dimensional Eigenfunctions for u_1



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$m = 1, 2, ...$ $\quad Y_m = b_m sin(\frac{m\pi y}{H})$ $\quad \lambda_{y_m} = (\frac{m\pi}{H})^2$

Two-Dimensional Eigenfunctions for u_1

$$\Phi_{nm} = sin(\frac{n\pi x}{L})sin(\frac{m\pi y}{H})$$

$$\lambda_{nm} = (\frac{n\pi}{L})^2 + (\frac{m\pi}{H})^2$$

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=> $$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} - b_{nm}\lambda_{nm} sin(\frac{n\pi x}{L})sin(\frac{m\pi y}{H}) = Q$$

Finishing Two-Dimensional Eigenfunctions

$$u_1 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{nm} sin(\frac{n\pi x}{L}) sin(\frac{m\pi y}{H})$$

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Finishing Two -Dimensional Eigenfunctions

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All we have left is coefficients

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Orthogonality(3 cases): s≠n/t≠m, s=n≠0/t=m≠0, s=n=0/t=m=0

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Finishing Two -Dimensional Eigenfunctions

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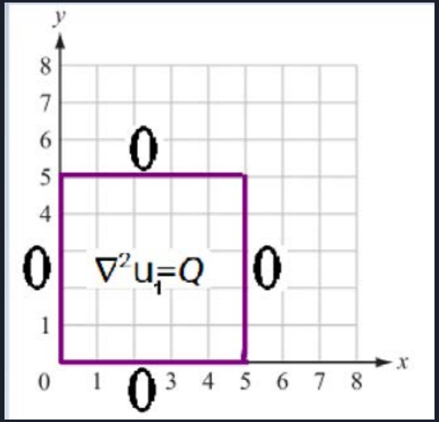
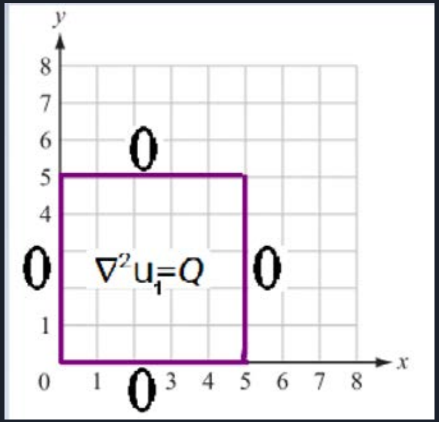
$n = 1, 2, ...$ $m = 1, 2, ...$

Finishing Solutions to u_1



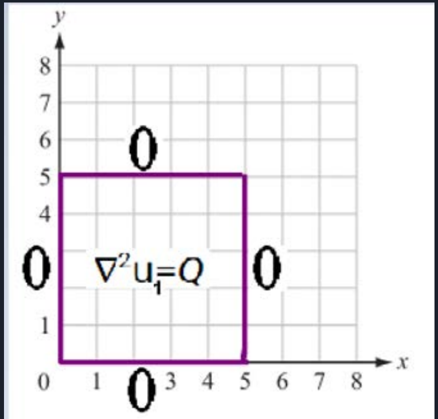
Finishing Solutions to u_1

- We now have our solutions for u_1



Finishing Solutions to u_1

- We now have our solutions for u_1
- Both One and Two-Dimensional



Finishing Solutions to u_1

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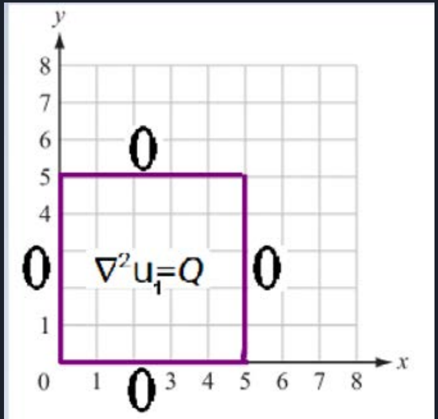
- $u = u_1 + u_2$



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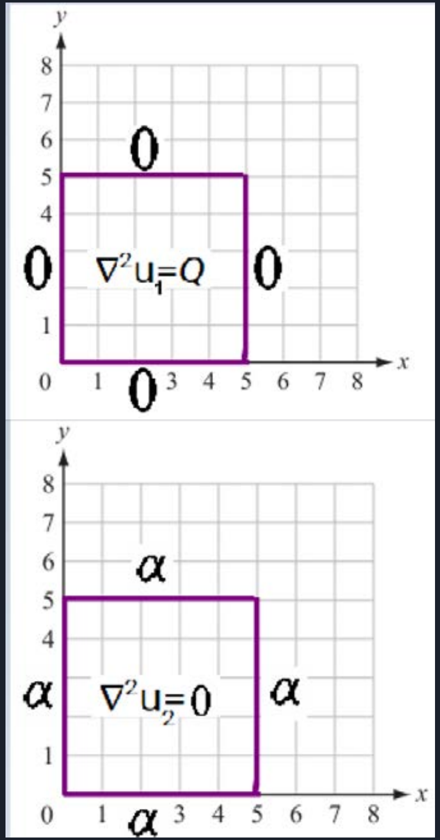
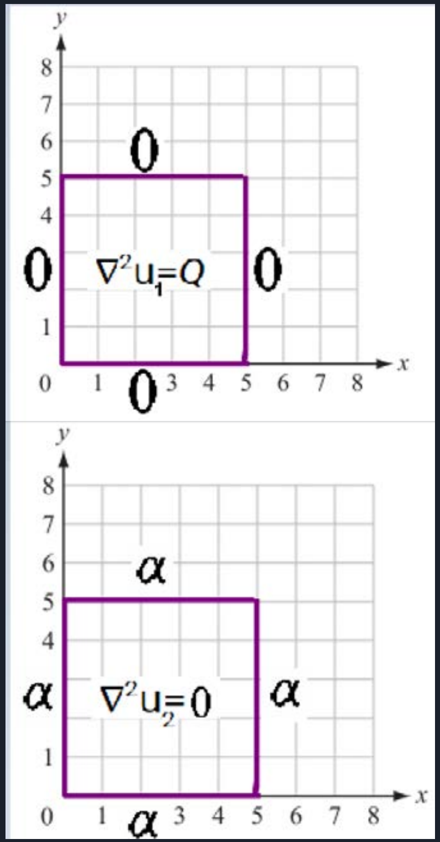
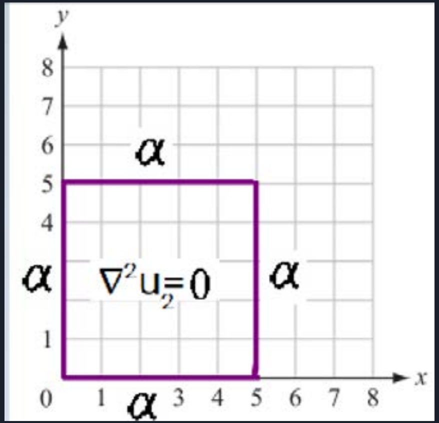


Finishing Solutions to u_1

- We now have our solutions for u_1
- Both One and Two-Dimensional
- $u = u_1 + u_2$
- u_2 left
- Thankfully, similar

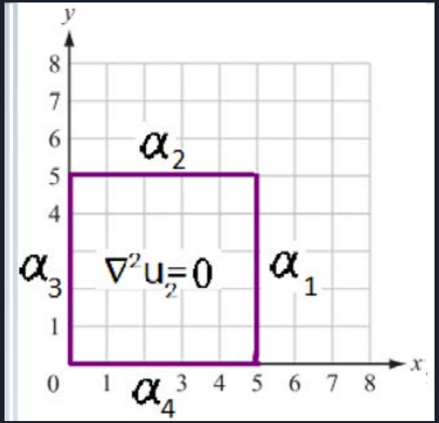


Solving $\Delta u_2 = 0$, $u_2 = \boldsymbol{\alpha}$ on Boundary



Solving $\Delta u_2 = 0$, $u_2 = \boldsymbol{\alpha}$ on Boundary

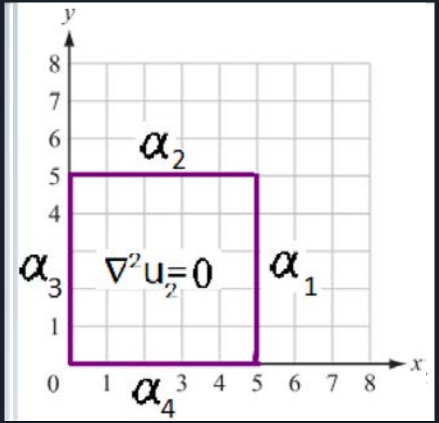
- To identify different edges



Solving $\Delta u_2 = 0$, $u_2 = \boldsymbol{\alpha}$ on Boundary

- To identify different edges
- To deal with nonhomogeneous boundary:

Let $u_2(x,y) = v(x,y) + w(x,y)$



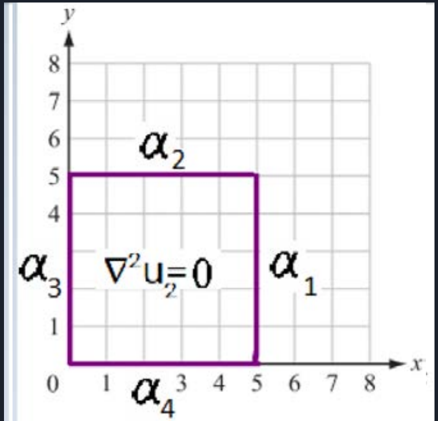
Solving $\Delta u_2 = 0$, $u_2 = \boldsymbol{\alpha}$ on Boundary

- To identify different edges
- To deal with nonhomogeneous boundary:

  Let $u_2(x,y) = v(x,y) + w(x,y)$

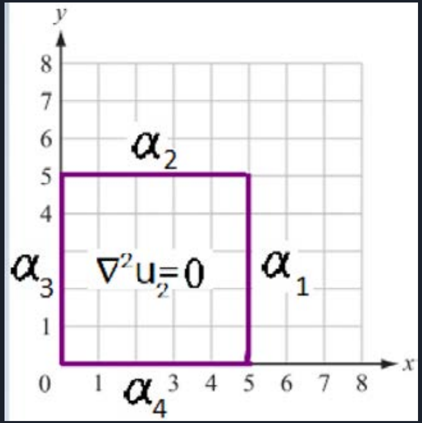
- Where $v(x, y)$ represents boundary

  and $w(x,y) = 0$ on boundary



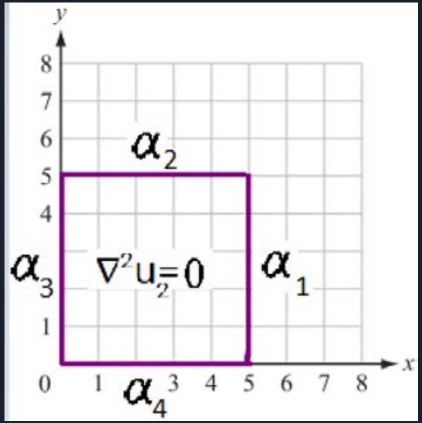
Solving $\Delta u_2 = 0$, $u_2 = \boldsymbol{\alpha}$ on Boundary

- $u_2(x,y) = v(x,y) + w(x,y)$



Solving $\Delta u_2 = 0$, $u_2 = \boldsymbol{\alpha}$ on Boundary

- $u_2(x,y) = v(x,y) + w(x,y)$
- $\Delta u_2 = u_{2xx} + u_{2yy}$



Solving $\Delta u_2 = 0$, $u_2 = \alpha$ on Boundary

- $u_2(x,y) = v(x,y) + w(x,y)$
- $\Delta u_2 = u_{2xx} + u_{2yy}$
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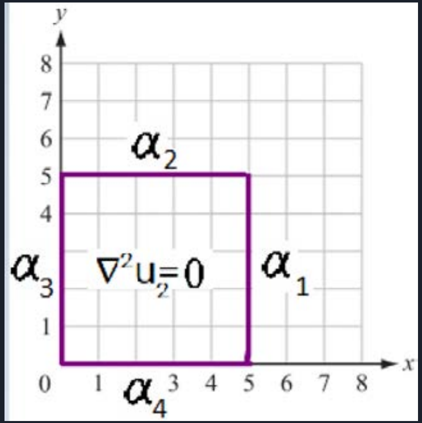

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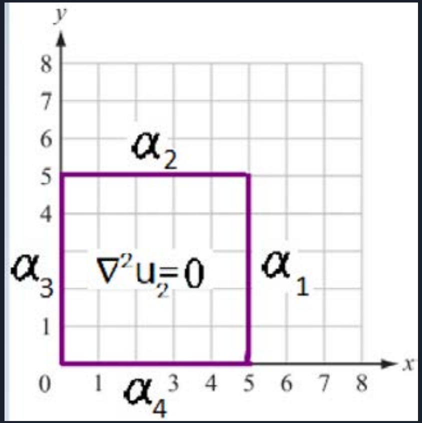
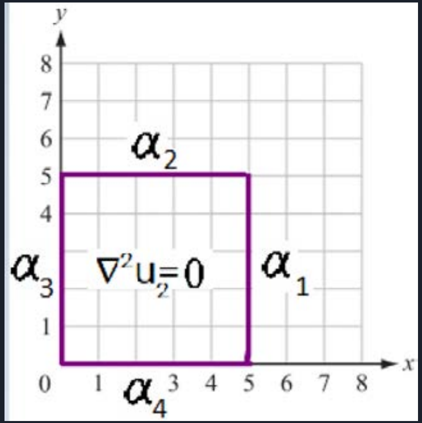
- $\Delta u_2 = u_{2xx} + u_{2yy}$
- $u_{2xx} = v_{xx} + w_{xx}$
- $u_{2yy} = v_{yy} + w_{yy}$
- $\Delta u_2 = w_{xx} + w_{yy} + v_{xx} + v_{yy}$



Solving $\Delta u_2 = 0$, $u_2 = \boldsymbol{\alpha}$ on Boundary

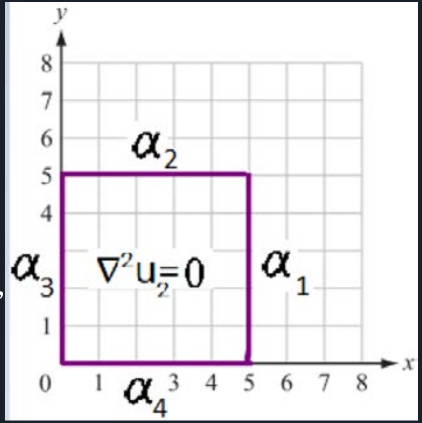
- $u_2(x,y) = v(x,y) + w(x,y)$
- $\Delta u_2 = u_{2xx} + u_{2yy}$
- $u_{2xx} = v_{xx} + w_{xx}$
- $u_{2yy} = v_{yy} + w_{yy}$
- $\Delta u_2 = w_{xx} + w_{yy} + v_{xx} + v_{yy}$
- $v_{xx} = v_{yy} = 0$



Solving v(x,y)

$v_{xx} = v_{yy} = 0$

Note: $\boldsymbol{\alpha}_{1,3} = \boldsymbol{\alpha}(y)_{1,}$

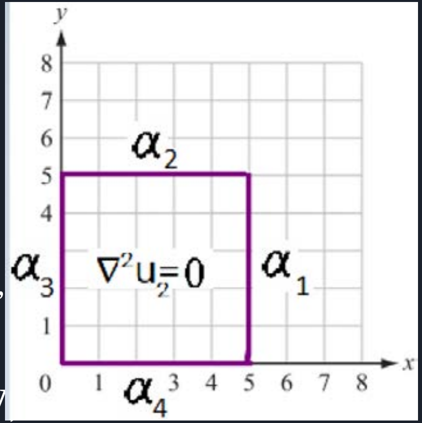


Solving v(x,y)

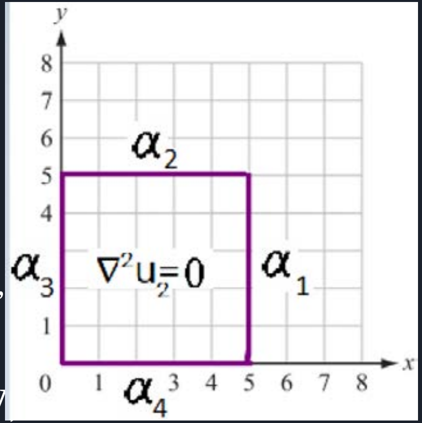
$v_{xx} = v_{yy} = 0$

Note: $\boldsymbol{\alpha}_{1,3} = \boldsymbol{\alpha}(y)_{1,}$

$v(x,y)_{xx} = 0$ $\quad$ $v(x,y)_x = f(y)$ $\quad$ $v(x,y) = f(y)x + g(y$



Solving v(x,y)



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Solving v(x,y)

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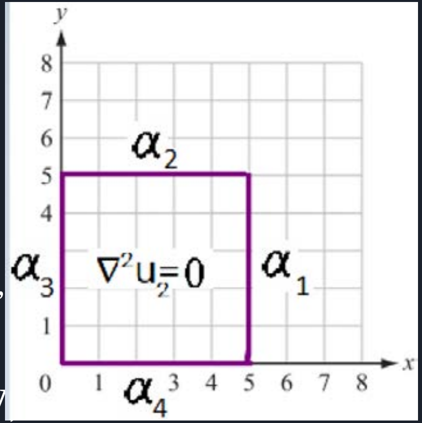

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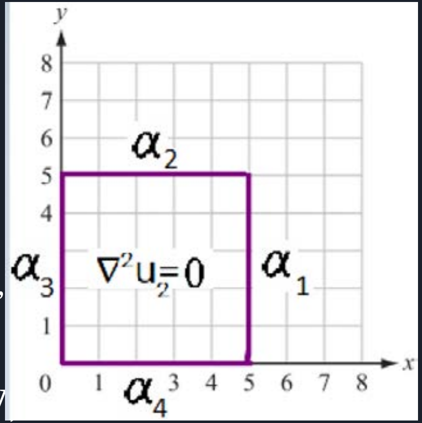
$v(0,y) = g(y) = \boldsymbol{\alpha}_3$ $\quad$ $v(L,y) = f(y)L + \boldsymbol{\alpha}_3 = \boldsymbol{\alpha}_1$

$$f(y) = \frac{\alpha_1 - \alpha_3}{L}$$

Solving v(x,y)

$v_{xx} = v_{yy} = 0$

Note: $\boldsymbol{\alpha}_{1,3} = \boldsymbol{\alpha}(y)_{1,}$



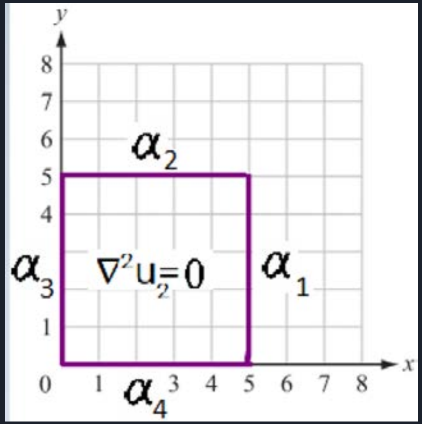
$v(x,y)_{xx} = 0$ $\quad v(x,y)_x = f(y)$ $\quad v(x,y) = f(y)x + g(y$

$v(0,y) = g(y) = \boldsymbol{\alpha}_3$ $\quad v(L,y) = f(y)L + \boldsymbol{\alpha}_3 = \boldsymbol{\alpha}_1$

$$f(y) = \frac{\alpha_1 - \alpha_3}{L} \qquad v = \left(\frac{\alpha_1 - \alpha_3}{L}\right)x + \alpha_3$$

Solving v(x,y)

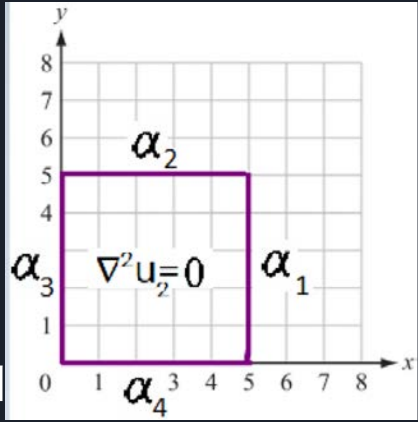
Similarly for v_{yy}:



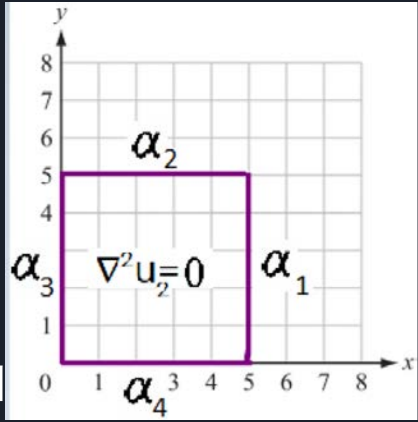
Solving v(x,y)

Similarly for v_{yy}:

$v(x,y)_{yy} = 0$ $v(x,y)_y = h(x)$ $v(x,y) = h(x)y + q$



Solving v(x,y)



Similarly for v_{yy}:

$v(x,y)_{yy} = 0$ $v(x,y)_y = h(x)$ $v(x,y) = h(x)y + q$

$v(x,0) = q(x) = \alpha_4$

Solving v(x,y)

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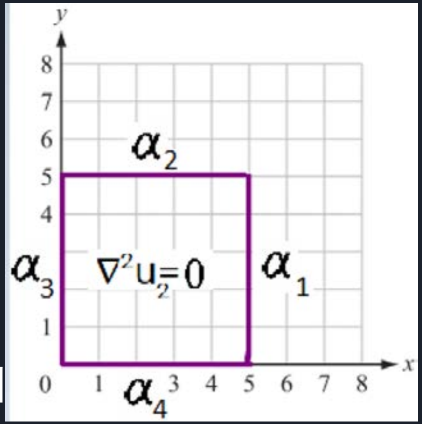
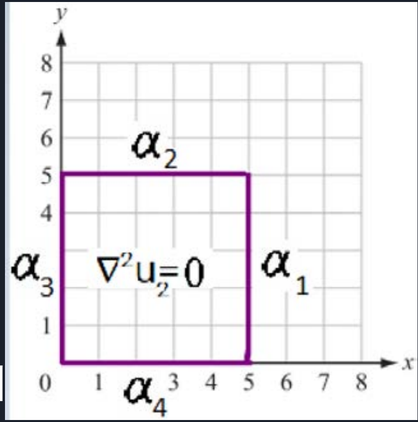


$v(x,0) = q(x) = \boldsymbol{\alpha}_4$ $\quad$ $v(H,y) = h(x)H + \boldsymbol{\alpha}_4 = \boldsymbol{\alpha}_2$

$$h(x) = \frac{\alpha_2 - \alpha_4}{H}$$

Solving v(x,y)



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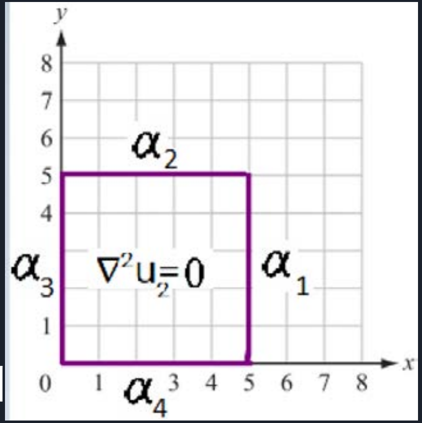
$$h(x) = \frac{\alpha_2 - \alpha_4}{H}$$

$$v = \left(\frac{\alpha_2 - \alpha_4}{H}\right)y + \alpha_4$$

Solving v(x,y)

Similarly for v_{yy}:

$v(x,y)_{yy} = 0$ $v(x,y)_y = h(x)$ $v(x,y) = h(x)y + q$



$v(x,0) = q(x) = \boldsymbol{\alpha}_4$ $v(H,y) = h(x)H + \boldsymbol{\alpha}_4 = \boldsymbol{\alpha}_2$

$$h(x) = \frac{\alpha_2 - \alpha_4}{H}$$

$$v = \left(\frac{\alpha_2 - \alpha_4}{H}\right)y + \alpha_4$$

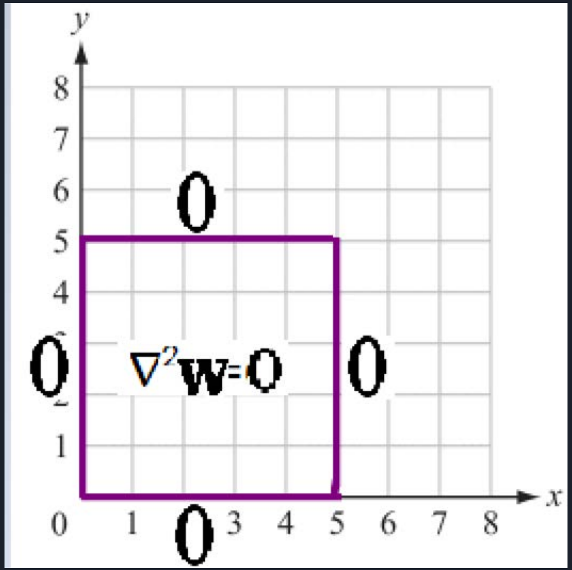
Add solution of v_{xx}: $$v(x,y) = \left(\frac{\alpha_1 - \alpha_3}{L}\right)x + \left(\frac{\alpha_2 - \alpha_4}{H}\right)y + \alpha_3 + \alpha_4$$

Solving w(x,y)

- $\Delta w = w_{xx} + w_{yy} = 0$, $w(x,y)=0$ on boundary

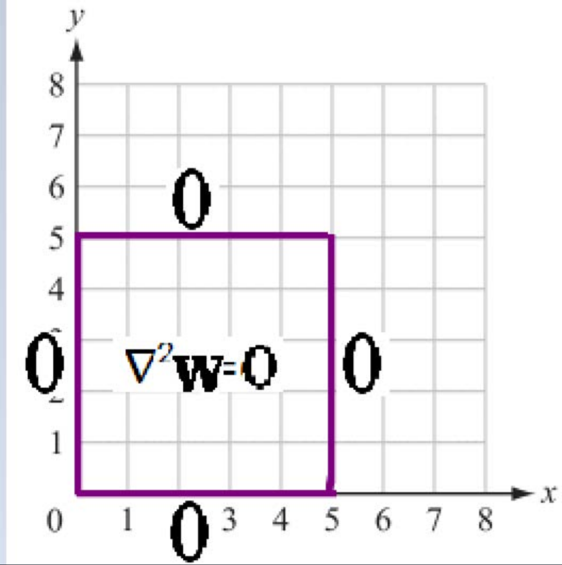
Solving w(x,y)

- $\Delta w = w_{xx} + w_{yy} = 0$, $w(x,y)=0$ on boundary



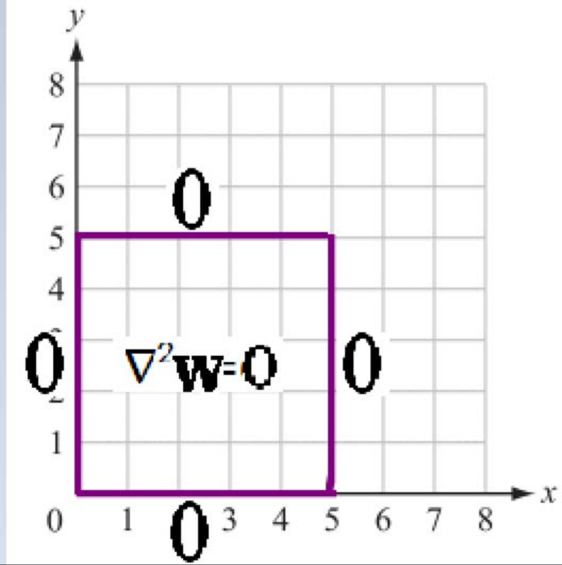
Solving w(x,y)

- $\Delta w = w_{xx} + w_{yy} = 0$, w(x,y)=0 on boundary
- Reminder: Time-independent



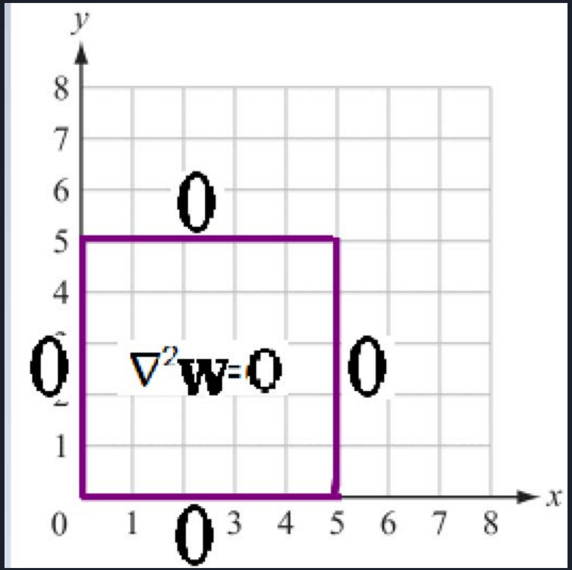
Solving w(x,y)

- $\Delta w = w_{xx} + w_{yy} = 0$, $w(x,y)=0$ on boundary
- Reminder: Time-independent
- The plate is 0 everywhere



Solving w(x,y)

- $\Delta w = w_{xx} + w_{yy} = 0$, w(x,y)=0 on boundary
- Reminder: Time-independent
- The plate is 0 everywhere
- w(x,y) = 0



Finishing u_2

- $u_2(x,y) = v(x,y) + w(x,y)$

Finishing u_2

- $u_2(x,y) = v(x,y) + w(x,y)$
- Substitution: $u_2(x,y) = (\frac{\alpha_1 - \alpha_3}{L})x + (\frac{\alpha_2 - \alpha_4}{H})y + \alpha_3 + \alpha_4$
- We now have the solutions to u_1 and u_2

Concluding Poisson's Equation

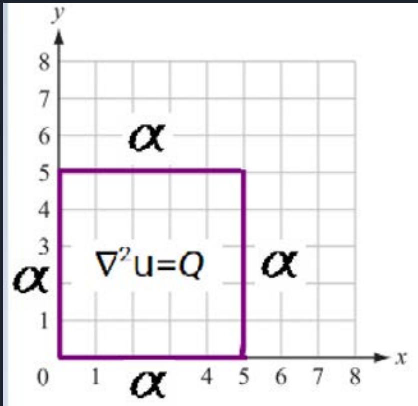
Δu = Q, u = **α** on boundary

$u = u_1 + u_2 = \sum_{n=1}^{\infty} b_n(y) \sin(\frac{n\pi x}{L}) \quad (\frac{\alpha_1 - \alpha_3}{L})x + (\frac{\alpha_2 - \alpha_4}{H})y + \alpha_3 + \alpha_4$

Where

$$b_n(y) = \sinh(\frac{n\pi[H-y]}{L}) \int_0^y q_n(\xi) \sinh(\frac{n\pi\xi}{L}) d\xi$$
$$+ \sinh(\frac{n\pi y}{L}) \int_y^H q_n(\xi) \sinh(\frac{n\pi(H-\xi)}{L}) d\xi$$

For one-dimensional



Concluding Poisson's Equation

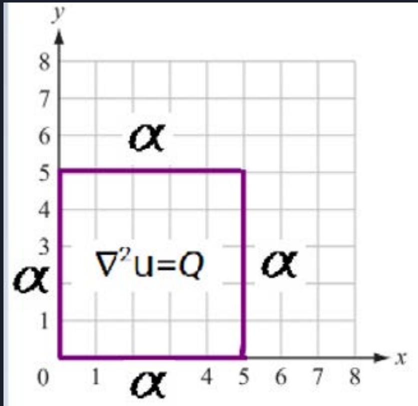
$\Delta u = Q$, $u = \alpha$ on boundary

$$u = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm} sin(\frac{n\pi x}{L}) sin(\frac{m\pi y}{H}) \quad (\frac{\alpha_1 - \alpha_3}{L})x + (\frac{\alpha_2 - \alpha_4}{H})y + \alpha_3 + \alpha_4$$

Where

$$b_{nm} = \frac{\int_0^H \int_0^L Q sin(\frac{n\pi x}{L}) sin(\frac{m\pi y}{H}) dxdy}{-\lambda_{nm} \int_0^H \int_0^L sin^2(\frac{n\pi x}{L}) sin^2(\frac{m\pi y}{H}) dxdy}$$

For two-dimensional



Finishing u_2

- $u_2(x,y) = v(x,y) + w(x,y)$
- Substitution: $u_2(x,y) = (\frac{\alpha_1 - \alpha_3}{L})x + (\frac{\alpha_2 - \alpha_4}{H})y + \alpha_3 + \alpha_4$

In Summary

- Purpose of Poisson’s Equation
- Solved Poisson’s Equation
- Nonhomogeneous Internal and boundaries
- One and Two dimensional ways
- Separation of Variables
- Orthogonality